# PROOF BY MATHEMATICAL INDUCTION: PROFESSIONAL PRACTICE FOR SECONDARY TEACHERS

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Mathematical induction is a proof technique that can be applied to establish the veracity of mathematical statements. This professional practice paper offers insight into mathematical induction as it pertains to the Australian Curriculum: Mathematics (ACMSM065, ACMSM066) and implications for how secondary teachers might approach this technique to students. In particular, literature on proof – and specifically, mathematical induction – will be presented, and several worked examples will outline the key steps involved in solving problems. After various teaching and learning caveats have been explored, the paper will conclude with some mathematical induction example problems that can be used in the secondary classroom.

## Introduction

A significant amount of mathematics involves the examination of patterns. Many of these patterns are concerned with generalisations about sequences and series. Mathematical induction is a method of proof argument that is based in recursion, and it is used for proving conjectures which claim that a certain statement is true for integer values of some variable. One idea that has been used to illustrate this method is to imagine a number of dominoes lined up in a row (Peressini et al., 1998). These authors suggest that for each integer $k \geq 1$, if the $k$th domino falls over then it will cause the ($k$+1)st domino to fall over as well. Furthermore, it could be argued specifically that if the first domino is pushed over, then all remaining dominoes would also fall.

If we suppose that for each positive integer $n$, $S(n)$ is a statement written in terms of $n$, then the principle of mathematical induction can be explained generally in two steps:

1. If $S(1)$ is true, and
2. for all integers $k \geq 1$, the assumption that $S(k)$ is true implies that $S(k+1)$ is true, then $S(n)$ is true for all positive integers $n$.

In other words, we commence the proof method through a verification of Step 1 (the *Initial Step*), or by pushing over the first domino. Then, we assume that $S(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$, known as the inductive assumption. In Step 2 (the *Base Induction Step*) we show that the supposition that $S(k)$ is true implies that $S(k+1)$ is true. Compared with the domino line-up, Step 2 corresponds to the assumption that if the $k$th domino falls then so will the $(k+1)$st domino.

## The Importance of Proof in Mathematics Education

Mathematical proof involves following a logical way to explain a hypothesis and to offer a cogent explanation of how deductive reasoning has been used to reach a conclusion. (Hanna, 1995; Tall, 1998). During the proving process, proofs require us to create "a sequence of steps, where each step follows logically from an earlier part of the proof where the last line is the statement being proved" (Garnier & Taylor, 2010, p. 50). The concept of proof is considered to be central to the discipline of mathematics, and because of this centrality, scholars have argued that proof should feature prominently in mathematics education (Ball et al., 2002; Baştürk, 2010; Siemon et al., 2015). Specifically, proof is recognised as an essential tool for promoting mathematical understanding in students (Ball et al., 2002; Reid, 2011) and for providing educators with insight about how students learn mathematics (Wilkerson-Jerde & Wilensky, 2011). Güler (2016) proposed that proof is important in mathematics education for various reasons, in that it: improves skills in problem solving, persuasive argumentation, reasoning, creativity and mathematical thinking. Moreover, proof forms the basis of mathematics, enables mathematical communication to transpire, and prevents rote learning of information.

## Mathematical Induction

Mathematical induction is considered one of the most powerful tools for proving statements in discrete mathematics (Ashkenazi & Itzkovitch, 2014). While there is endless scope for the types of problems mathematical induction can be applied to, three popular 'types' of problems are used by teachers when teaching this type of mathematical proof to secondary students. These problem types include: General series, divisibility and implication. Each of these types will now be presented as a worked example.

### General series

Let us propose that we are interested in finding a general statement to explain the sum of $n$ consecutive odd integers starting at 1. If we tabulate our findings for the first 10 natural or counting numbers, and their partial sums, we have:

Table 1. Counting numbers and their sums, $1 \leq n \leq 10$.

| $n$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|---|---|---|
| $T_n$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 |
| $S_n$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |

It should be noted that the row $T_n$ represents the nth odd integer, and the row $S_n$ is the sum of the first $n$ odd integers. One interesting pattern that can be observed is that the last row of the table, $S_n$, shows all integers $n^2$ for $n \geq 1$. A cursory comparison between the three rows reveals that the sum of the first $n$ odd numbers appears to be the square of $n$. In making this statement, we have arrived at a conjecture – which is the first step in working towards a theorem – but we may not even know if the statement is true! The following worked example provides a precise mathematical statement of the result we are trying to prove.

## Worked example 1: General series

Prove by mathematical induction that for all integers $n \geq 1$

$$S(n): \ 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

*Worked solution*

1. Initial step: We need to show that the conjecture is true for a small value of $n$, e.g. $n = 1$. Substituting this value into the series we have:

$1 = 1^2$
which is clearly true
∴ we have shown that $S(1)$ is true

2. Inductive Step: Here we assume the statement (inductive hypothesis)

$$S(k): \ 1 + 3 + 5 + \cdots + (2k-1) = k^2 \qquad (1)$$

is true for a fixed but arbitrary value of $k \geq 1$ and verify that the statement

$$S(k+1): \ 1 + 3 + 5 + \cdots + (2k-1) + [2(k+1)-1] = (k+1)^2 \qquad (2)$$

Looking back at (1), we can see that the series $1 + 3 + 5 + \cdots + (2k-1)$ exists in (2). We therefore substitute $k^2$ into (2) for $1 + 3 + 5 + \cdots + (2k-1)$, and algebraically rewrite the Left Hand Side (LHS) until it equals the Right Hand Side (RHS).

$$\begin{aligned} \text{LHS} &= 1 + 3 + 5 + \cdots + (2k-1) + [2(k+1)-1] \\ &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2 = \text{RHS} \end{aligned}$$

Conclusion: Because we have verified the initial and inductive steps we can conclude by induction that the statement

$$\mathrm{S(n)}: \ 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

is true for all integers $n \geq 1$.

## Worked example 2: Divisibility

Prove by mathematical induction that for all integers $n \geq 1$

$$S(n): \ 3^{2n} - 1 \text{ is divisible by } 8.$$

*Worked solution*

1. Initial step: We need to show that the statement $S(1)$ is true. Substituting $n = 1$ into the expression gives us:

$$3^{2(1)} - 1 = 3^2 - 1 = 9 - 1 = 8$$

which is clearly divisible by 8.

Therefore, $S(1)$ is true.

2. Inductive step: We assume that the statement (inductive hypothesis)

$$3^{2k} - 1 \text{ is divisible by } 8 \qquad (1)$$

is true for a fixed and arbitrary value *of* $k \geq 1$. We must verify that the statement

$$S(k+1): \; 3^{2(k+1)} - 1 \text{ is divisible by } 8$$

is true. Now, we manipulate the expression $3^{2(k+1)} - 1$ using algebraic rules until it becomes divisible by 8.

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1$$
$$= 3^{2k} \times 3^2 - 1$$
$$= 3^{2k}(9) - 1$$
$$= 3^{2k}(8+1) - 1$$
$$= 8 \times 3^{2k} + 3^{2k} - 1 \qquad (2)$$

Now because from (1) we have assumed that $3^{2k} - 1$ is divisible by 8, there are two terms which are divisible by 8 – one proven through clear algebra, and the other via an assumption from the inductive step. As such, both terms of (2) are divisible by 8 and therefore so is their sum. In other words, $S(k+1)$ is true.

## Worked example 3: Inequalities

Using mathematical induction, prove that for all integers $n \geq 3$

$$S(n): \; 2^n > 2n + 1$$

*Worked solution*

1. Initial step: We need to show that the statement $S(3)$ is true. Substituting $n=3$ into this expression gives:

$$2^3 > 2(3) + 1$$
$$8 > 7$$

which is clearly true

Therefore, $S(3)$ is true.

2. Inductive step: We assume that the statement (inductive hypothesis)

$$S(k): \; 2^k > 2k + 1 \qquad (1)$$

is true for a fixed and arbitrary value of $k \geq 3$. We must verify the statement

$$S(k+1):\ 2^{k+1} > 2(k+1)+1 \qquad (2)$$

We now manipulate both sides of (1) to transform it into (2). In other words, the inductive statement will be manipulated algebraically so the values of n = k have been transformed into n = k + 1. Once we have done this, by implication we will have shown that the statement will remain true for all values of k and the very next value after k. Ideally, the 'finished product' will look like:

$$2^{k+1} > 2(k+1)+1$$

Some annotations have been included on the RHS of the inequality to assist in following the steps in working out.

| | |
|---|---|
| $2^k \times 2 > 2(2k+1)$ | Multiply both sides by 2 |
| $2^{k+1} > 4k+2$ | Simplify |
| $2^{k+1} > 2k+2k+2$ | Re-express the RHS terms |
| $2^{k+1} > 2k+2+2k$ | Rearrange the RHS terms |
| $2^{k+1} > 2(k+1)+2k$ | Factorise the first two terms |

Now, as the original problem stated, $n \geq 3$ which implies that the LHS of the original statement $2n + 1 > 1$. In particular, if we substitute $n = 3$ into the LHS we obtain a value of 7, which is clearly greater than 1. As such we can create a concatenated inequality statement:

$$2^{k+1} > 2(k+1)+2k > 2(k+1)+1$$
$$\therefore 2^{k+1} > 2(k+1)+1$$

In this way, the inductive step $S(k)$ has implied $S(k+1)$ is true.

## Some caveats associated with mathematical induction

A review of literature on mathematical induction reveals that this method is difficult to teach for a variety of reasons (Ashkenazi & Itzkovitch, 2014; Harel, 2002; Stylianides et al., 2007). To commence, Ashkenazi and Itzkovitch (2014) contended that although secondary school and university students can successfully apply this proof method to statements of the kind they are accustomed to, they do not understand the correctness of the proof. Put another way, these authors suggest that most students learn how to use the method mechanically; such learning does not foster a deep understanding of the correctness of the method and ultimately contributes to a failure to solve problems of a different style (Ashkenazi & Itzkovitch, 2014). Echoing the contention of these authors contention, both Harel (2002) and Stylianides et al. (2007) asserted that undergraduate university students often display both a fragile knowledge on mathematical induction and a propensity to follow the steps without understanding what they are doing. In his analysis of students' attempts at mathematical induction, Harel (2002) further identified two specific difficulties experienced by students. First, students tended to consider mathematical induction as a case of circular reasoning as they believe that the proof assumes $S(n)$ is true for all positive integers. Second, students demonstrated a belief that the general argument for mathematical induction can be derived from a number of particular cases, rather than proving for all cases.

### Divisibility

An alternative method that can be used to prove induction divisibility problems (such as Worked Example 2) requires the use of two assumptions. Because the strength of a mathematical argument relies on the extent to which assumptions are minimised, the method shown below should be treated cautiously and avoided. If we recommence Worked Example 2 at the Inductive Step, it could be written that:

$$3^{2k} - 1 = 8A \text{ for some integer } A,\, A \geq 1$$

We can rearrange this inductive assumption as $3^{2k} = 8A + 1$ (1), which will be used when manipulating the statement $S(k+1)$. Herein:

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \times 3^2 - 1 = 9 \times 3^{2k} - 1$$

We now substitute (1): $= 9(8A + 1) - 1 = 72A + 9 - 1 = 72A + 8$

$= 8(9A + 1)$ which is clearly a multiple of 8

Having completed the necessary algebraic manipulations to reach a final statement which is divisible by 8, we are able to conclude that the conjecture is indeed true. However, looking back at the Inductive Step, we assumed that not only was the conjecture true for $k \geq 1$ but we also assumed that it was equal to a 8A (a multiple of 8) for $A \geq 1$. As such, the inductive assumption itself rested upon an assumption, which is a practice that should be avoided. Rather, to fulfil the logical steps of the proof we need to actually use the inductive assumption of the proof (i.e. $3^{2k} - 1$) in the final stages and not a substitute.

## Conclusion

The purpose of this paper was to offer insight to educators about proof by mathematical induction as it pertains to the Australian Curriculum: Mathematics. In particular, this method of proof has been outlined in a step-by-step fashion, and some worked examples have been offered to amplify these steps and the theoretical approach overall. Additionally, a cursory review of literature has revealed how scholars have championed the place of proof in a mathematics curriculum. In a study where mathematics professors were asked to evaluate and score undergraduate university students' completion of proofs (an example of mathematical induction was Task 4), these professors acknowledged that the most important characteristics of a well-written proof are logical correctness, clarity, fluency, and demonstration of understanding of the proof (Moore, 2016). It is the author's hope that this paper will be useful to mathematics educators within Australia – and perhaps internationally – as they model to secondary students how to apply the principles of mathematical induction to statements. Moreover, it is hoped that as students strive to master those characteristics of well-written proofs, their efforts will be underscored by a demonstration of procedural understanding.

## Examples to Try With Secondary Students

Use mathematical induction to prove that for all positive integers $n$:

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Use mathematical induction to prove that for all positive integers $n$:

4. $5^n + 3$ is divisible by 4
5. $3^{4n} - 1$ is divisible by 80
6. $4^n - 1$ is divisible by 3

Use mathematical induction to prove the following statements for all natural numbers $n \geq 5$:

7. $2^n > n^2$
8. $4n < 2^n$
9. $1 \times 2 \times 3 \times \ldots \times (n-1) > 2^n$

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