Mathematical Logic

Part One

Announcements

- Problem Session tonight from 7:00 – 7:50 in 380-380X.
  - Optional, but highly recommended!
- Problem Set 3 Checkpoint due right now.
- 2× Handouts
  - Problem Set 3 Checkpoint Solutions
  - Diagonalization
- Problem Set 2 Solutions distributed at end of class.

Office Hours

- We finally have stable office hours locations!
- Website will be updated soon with details.

An Important Question

How do we formalize the logic we've been using in our proofs?

Where We're Going

- **Propositional Logic** (Today)
  - Basic logical connectives.
  - Truth tables.
  - Logical equivalences.
- **First-Order Logic** (Today / Wednesday)
  - Reasoning about properties of multiple objects.

Propositional Logic

A **proposition** is a statement that is, by itself, either true or false.

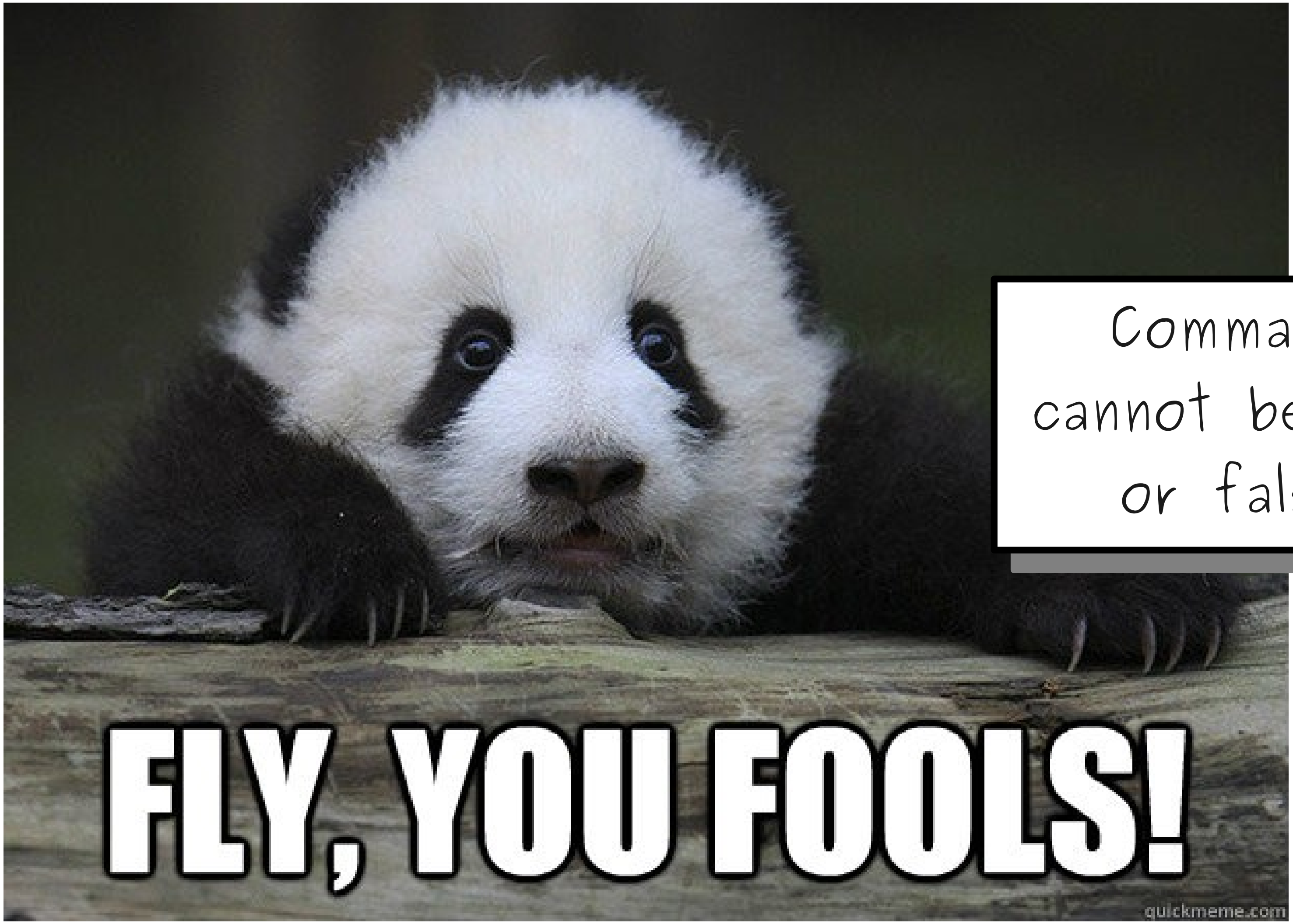
Some Sample Propositions

- Puppies are cuter than kittens.
- Kittens are cuter than puppies.
- Usain Bolt can outrun everyone in this room.
- CS103 is useful for cocktail parties.
- This is the last entry on this list.

More Propositions

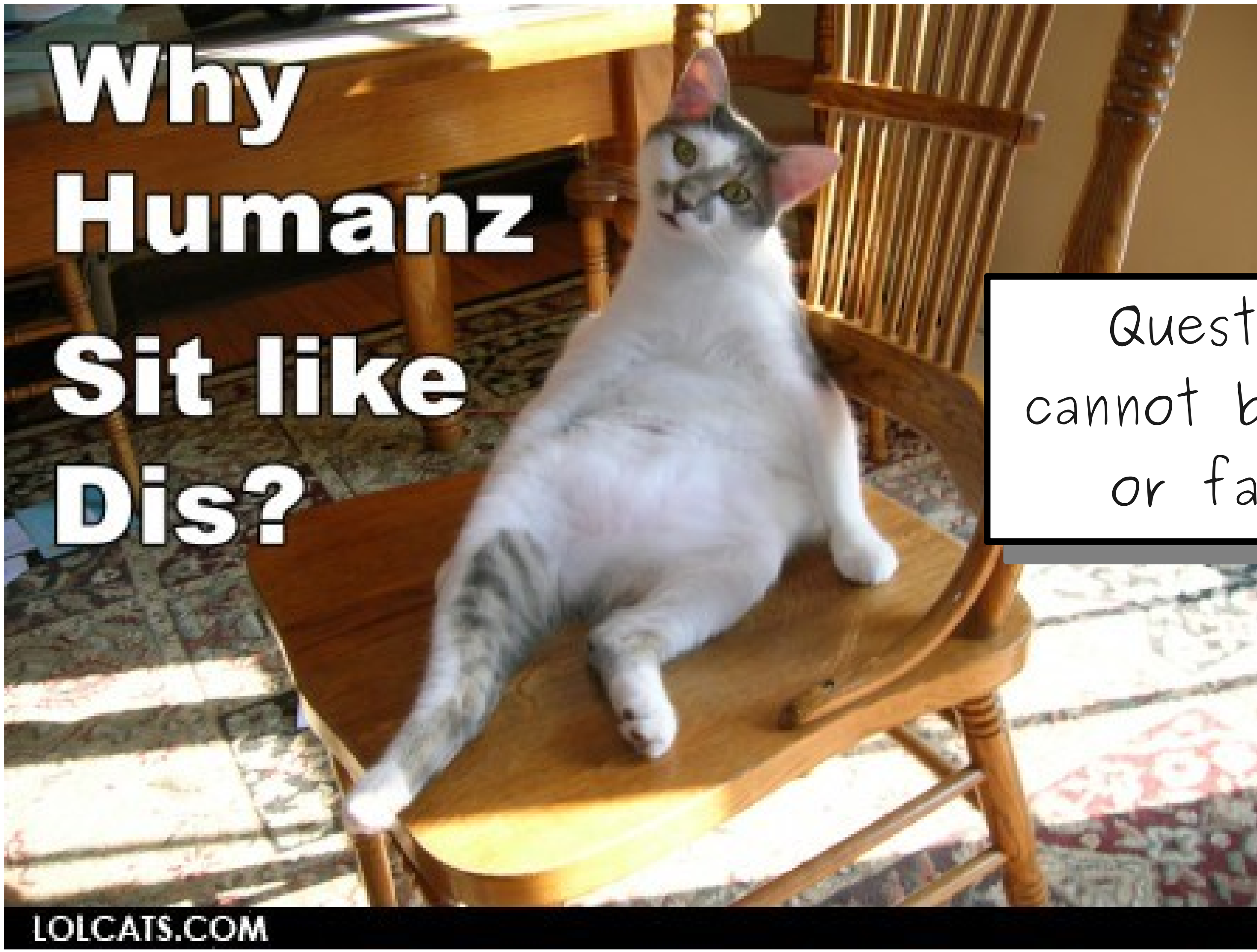
- I'm a single lady.
- This place about to blow.
- Party rock is in the house tonight.
- We can dance if we want to.
- We can leave your friends behind.

Things That Aren't Propositions



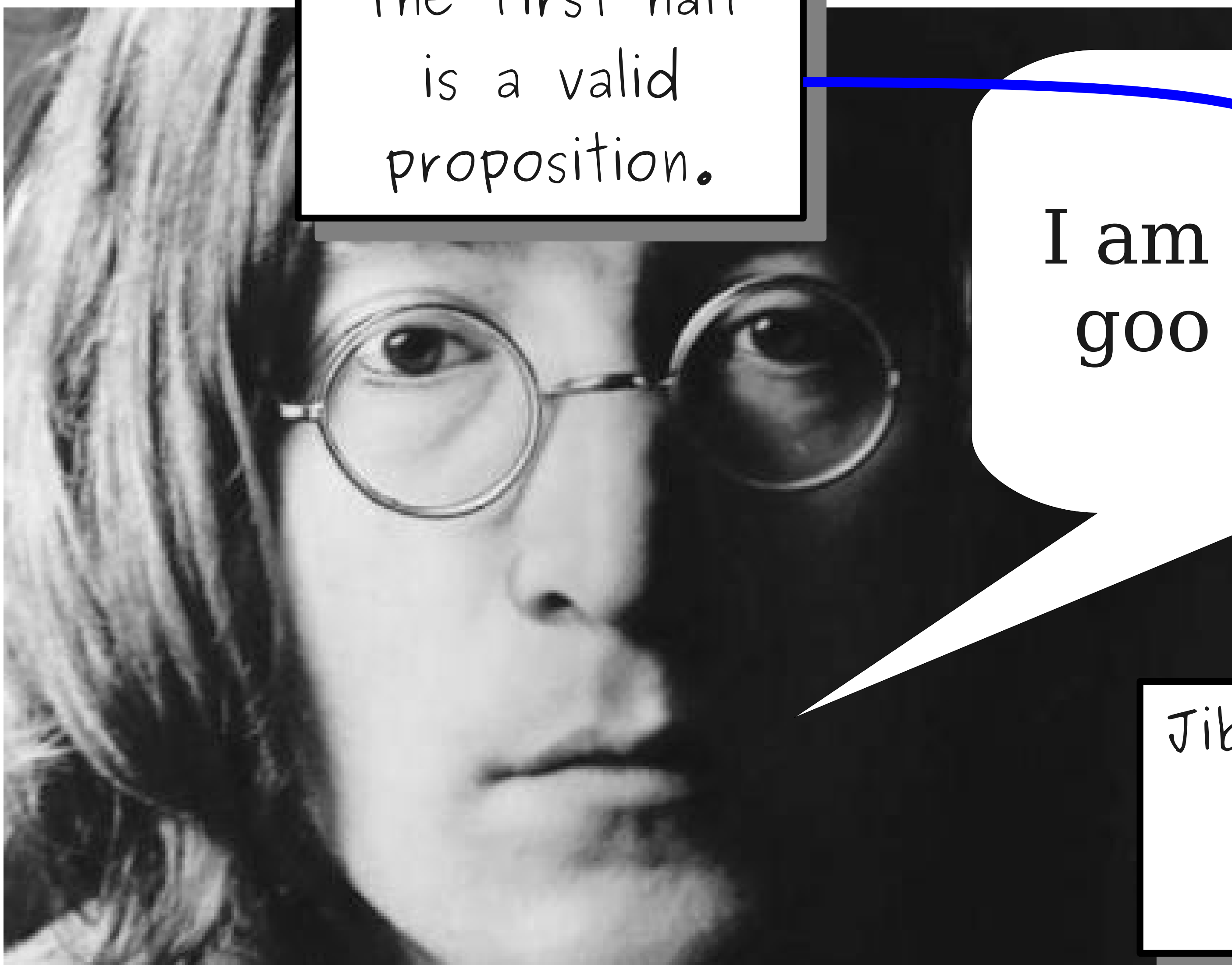
Commands cannot be true or false.

Things That Aren't Propositions



Questions cannot be true or false.

Things That Aren't Propositions



Propositional Logic

- **Propositional logic** is a mathematical system for reasoning about propositions and how they relate to one another.
- Propositional logic enables us to
  - Formally encode how the truth of various propositions influences the truth of other propositions.
  - Determine if certain combinations of propositions are always, sometimes, or never true.
  - Determine whether certain combinations of propositions logically entail other combinations.

Variables and Connectives

- Propositional logic is a formal mathematical system whose syntax is rigidly specified.
- Every statement in propositional logic consists of **propositional variables** combined via **logical connectives**.
  - Each variable represents some proposition, such as “You wanted it” or “You should have put a ring on it.”
  - Connectives encode how propositions are related, such as “If you wanted it, you should have put a ring on it.”

Propositional Variables

- Each proposition will be represented by a **propositional variable**.
- Propositional variables are usually represented as lower-case letters, such as p, q, r, s, etc.
  - If we need more, we can use subscripts: p_1, p_2, etc.
- Each variable can take one one of two values: true or false.

Logical Connectives

- **Logical NOT: $\neg p$**
  - Read "**not** p"
  - $\neg p$ is true if and only if p is false.
  - Also called **logical negation**.
- **Logical AND: $p \wedge q$**
  - Read "p **and** q."
  - $p \wedge q$ is true if both p and q are true.
  - Also called **logical conjunction**.
- **Logical OR: $p \vee q$**
  - Read "p **or** q."
  - $p \vee q$ is true if at least one of p or q are true (inclusive OR)
  - Also called **logical disjunction**.

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables

p	q	$p \lor q$
F	F	F
F	T	T
T	F	T
T	T	T

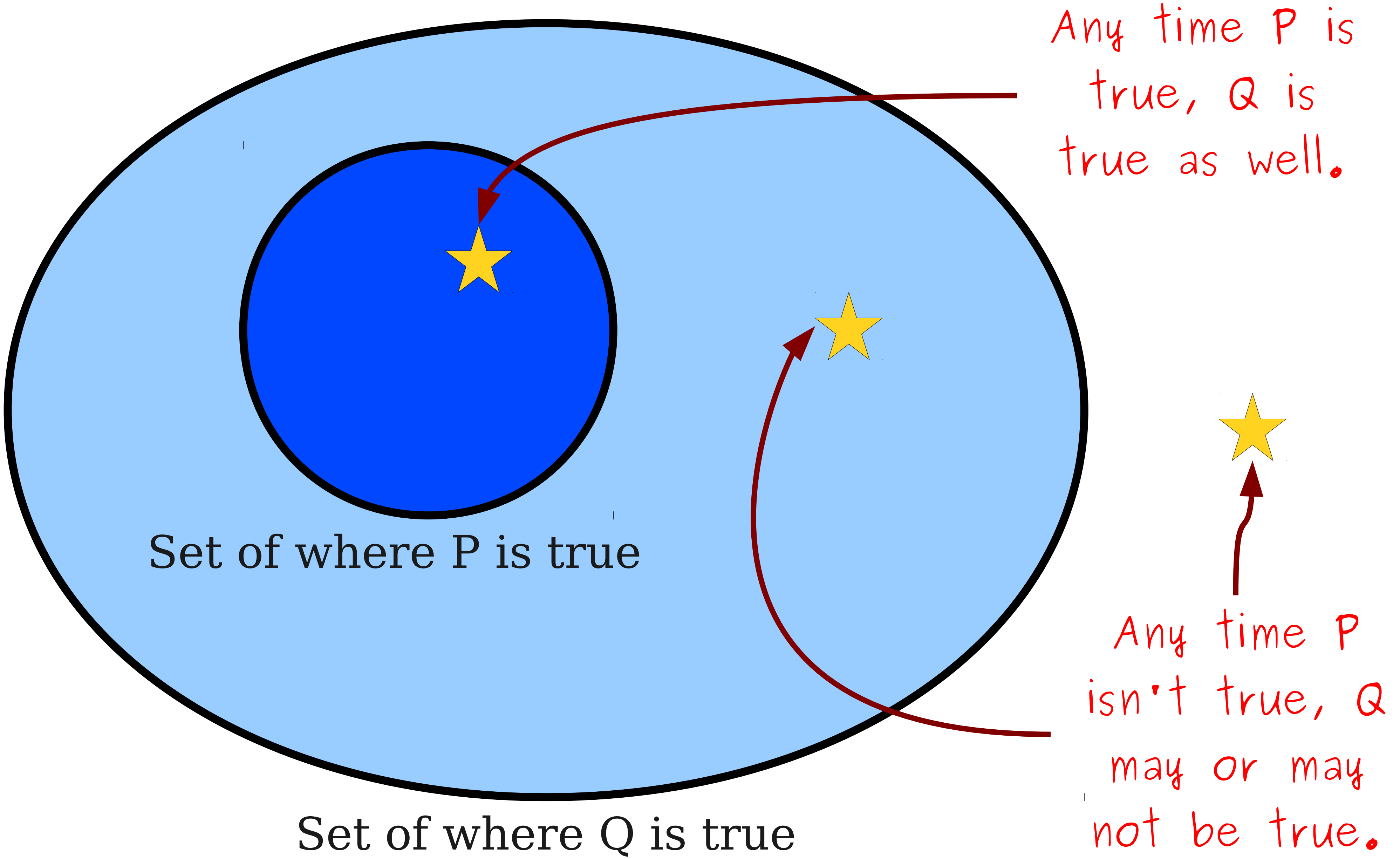
Truth Tables

p	$\neg p$
F	T
T	F

Implication

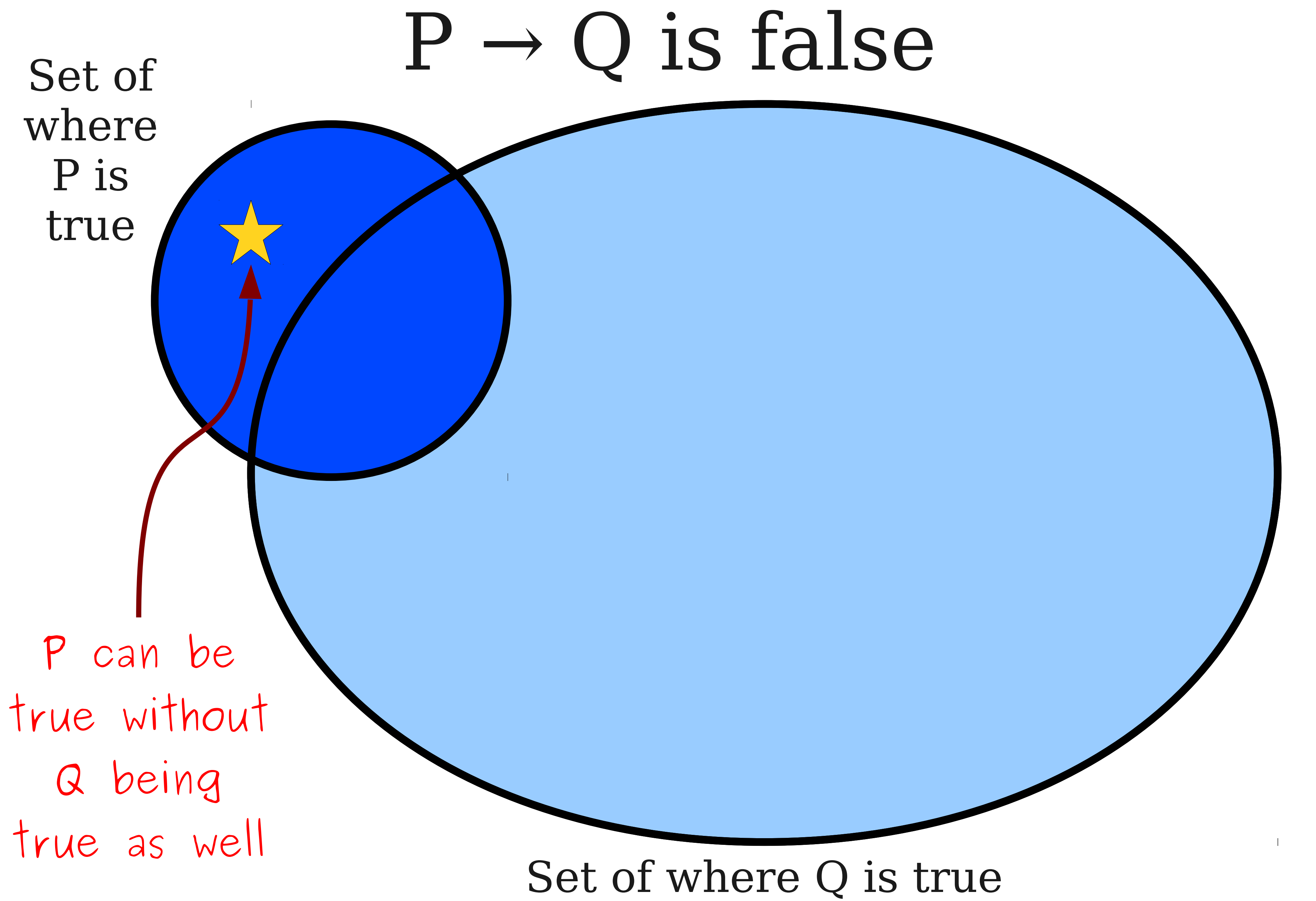
- An important connective is logical implication: $p \to q$.
- Recall: $p \to q$ means "if p is true, q is true as well."
- Recall: $p \to q$ says **nothing** about what happens if p is false.
- Recall: $p \to q$ says **nothing** about causality; it just says that if p is true, q will be true as well.

Implication, Diagrammatically



When p Does Not Imply q

- $p \to q$ means "if p is true, q is true as well."
- Recall: The **only way** for $p \to q$ to be false is if we know that p is true but q is false.
- Rationale:
  - If p is false, $p \to q$ doesn't guarantee anything. It's true, but it's not **meaningful**.
  - If p is true and q is true, then the statement "if p is true, then q is also true" is itself true.
  - If p is true and q is false, then the statement "if p is true, q is also true" is false.


P → Q is false
Set of where P is true
P can be true without Q being true as well
Set of where Q is true

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

The Biconditional

- The **biconditional** connective $p \leftrightarrow q$ is read "p if and only if q."
- Intuitively, either both p and q are true, or neither of them are.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

One interpretation of $\leftrightarrow$ is to think of it as equality: the two propositions must have equal truth values.

True and False

- There are two more “connectives” to speak of: true and false.
- The symbol $\top$ is a value that is always true.
- The symbol $\bot$ is value that is always false.
- These are often called connectives, though they don't connect anything.
  - (Or rather, they connect zero things.)

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to ((y \lor z) \to (x \lor (y \land z)))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Recap So Far

- A **propositional variable** is a variable that is either true or false.
- The **logical connectives** are
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

I won't be eaten by a velociraptor if there isn't a velociraptor outside my apartment.

$$\neg a \to \neg e$$

“p if q”

translates to

$$q \to p$$

It does **not** translate to

$$p \to q$$

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

If there is a velociraptor outside my apartment, but it can't open windows, I am not going to be eaten by a velociraptor.

$$a \land \neg b \to \neg e$$

“p, but q”

translates to

$$p \wedge q$$

Some Sample Propositions

a: There is a velociraptor outside my apartment.

b: Velociraptors can open windows.

c: I am in my apartment right now.

d: My apartment has windows.

e: I am going to be eaten by a velociraptor

I am only in my apartment when there are no velociraptors outside.

$$c \to \neg a$$

“p only when q”

translates to

$$p \rightarrow q$$

The Takeaway Point

- When translating into or out of propositional logic, be very careful not to get tripped up by nuances of the English language.
  - In fact, this is one of the reasons we have a symbolic notation in the first place!
- Many prepositions lead to counterintuitive translations; make sure to double-check yourself!

Logical Equivalence

More Elaborate Truth Tables

This gives the final truth value for the expression.

p	q	$p \land (p \to q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

Negations

- $p \wedge q$ is false if and only if $\neg(p \wedge q)$ is true.
- Intuitively, this is only possible if either p is false or q is false (or both!)
- In propositional logic, we can write this as $\neg p \vee \neg q$.
- How would we prove that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equivalent?
- **Idea**: Build truth tables for both expressions and confirm that they always agree.

Negating AND

p	q	$\neg(p \land q)$	
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p$	$\lor$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

These two statements are always the same!

Logical Equivalence

- If two propositional logic statements φ and ψ always have the same truth values as one another, they are called **logically equivalent**.
- We denote this by **φ ≡ ψ**.
- ≡ is **not** a connective. Connectives are a part of logic statements; ≡ is something used to describe logic statements.
  - It is part of the **metalanguage** rather than the **language**.
- If φ ≡ ψ, we can modify any propositional logic formula containing φ by replacing it with ψ.
  - This is not true when we talk about first-order logic; we'll see why later.

De Morgan's Laws

- Using truth tables, we concluded that

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

- We can also use truth tables to show that

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

- These two equivalences are called **De Morgan's Laws**.

More Negations

- When is $p \to q$ false?
- **Answer**: p must be true and q must be false.
- In propositional logic:

$$\boldsymbol{p \land \neg q}$$

- Is the following true?

$$\boldsymbol{\neg(p \to q) \equiv p \land \neg q}$$

Negating Implications

p	q	$\neg(p \to q)$	
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	$p \land \neg q$		
F	F	F	F	T
F	T	F	F	F
T	F	T	T	T
T	T	T	F	F

$$\neg(p \to q) \equiv p \land \neg q$$

An Important Observation

- We have just proven that

$$\boldsymbol{\neg(p \to q) \equiv p \land \neg q}$$

- If we negate both sides, we get that

$$\boldsymbol{p \to q \equiv \neg(p \land \neg q)}$$

- By De Morgan's laws:

$$p \to q \equiv \neg(p \land \neg q)$$

$$p \to q \equiv \neg p \lor \neg\neg q$$

$$p \to q \equiv \neg p \lor q$$

- Thus $\boldsymbol{p \to q \equiv \neg p \lor q}$

Another Idea

- We've just shown that $\neg(p \to q) \equiv p \land \neg q$.
- Is it also true that $\neg(p \to q) \equiv p \to \neg q$?
- Let's go check!

$$\neg(p \to q) \text{ and } p \to \neg q$$

p	q	$\neg$	$(p \to q)$
F	F	F	T
F	T	F	T
T	F	T	F
T	T	F	T

p	q	p	$\to$	$\neg q$
F	F	F	T	T
F	T	F	T	F
T	F	T	T	T
T	T	T	F	F

These are *not* the same thing!

To prove that $p \to q$ is false, do **not** prove $p \to \neg q$.

Instead, prove that $p \land \neg q$ is true.

Analyzing Proof Techniques

Proof by Contrapositive

- Recall that to prove that $p \to q$, we can also show that $\neg q \to \neg p$.
- Let's verify that $p \to q \equiv \neg q \to \neg p$.

The Contrapositive

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$\neg q$	$\to$	$\neg p$
F	F	T	T	T
F	T	F	T	T
T	F	T	F	F
T	T	F	T	F

$$p \to q \equiv \neg q \to \neg p$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x < 8 \land y < 8 \rightarrow x + y \neq 16$$

“If $x < 8$ and $y < 8$, then $x + y \neq 16$”

Theorem: If $x + y = 16$, then either $x \geq 8$ or $y \geq 8$.

Proof: By contrapositive. We prove that if $x < 8$ and $y < 8$, then $x + y \neq 16$. To see this, note that

$$\begin{aligned} x + y &< 8 + y \\ &< 8 + 8 \\ &= 16 \end{aligned}$$

So $x + y < 16$, so $x + y \neq 16$. ■

Why This Matters

- Propositional logic is a tool for reasoning about how various statements affect one another.
- To better understand how to prove a result, it often helps to translate what you're trying to prove into propositional logic first.
- **Note**: To truly reason about proofs, we need the more expressive power of **first-order logic**, which we'll talk about next time.

Proof by Contradiction

- The general structure of a proof by contradiction is
  - To show *p*, assume *p* is false.
  - Show that *p* being false implies something that cannot be true.
  - Conclude, therefore, that p is true.
- What does this look like in propositional logic?

$$(\neg p \to \bot) \to p$$

Proof by Contradiction

p	$(\neg p$	$\to$	$\bot)$	$\to$	p
F	T	F	F	T	F
T	F	T	F	T	T

This statement
is <u>always</u> true!

Tautologies

- A **tautology** is a statement that is always true.
- Examples:
  - $\top$
  - $p \lor \neg p$ (the **Law of the Excluded Middle**)
  - $\bot \to p$ (**vacuous truth**)
- Once a tautology has been proven, we can use that tautology anywhere.

Next Time

- **First-Order Logic**
  - How do we reason about multiple objects and their properties?