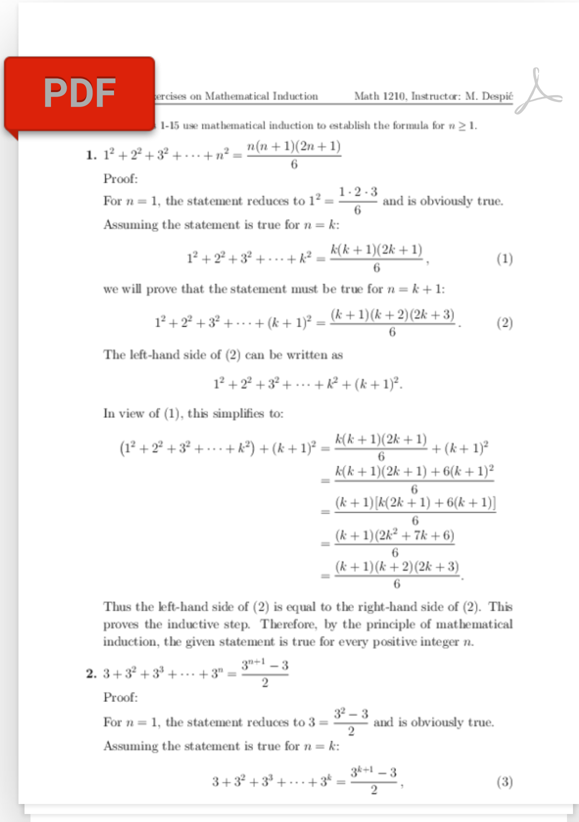

ercises on Mathematical Induction   Math 1210, Instructor: M. Despić

1-15 use mathematical induction to establish the formula for $n \ge 1$.

1. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof:

For $n = 1$, the statement reduces to $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}, \qquad (1)$$

we will prove that the statement must be true for $n = k + 1$:

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}. \qquad (2)$$

The left-hand side of (2) can be written as

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2.$$

In view of (1), this simplifies to:

$$\begin{aligned}(1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

Thus the left-hand side of (2) is equal to the right-hand side of (2). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

2. $3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}$

Proof:

For $n = 1$, the statement reduces to $3 = \frac{3^2 - 3}{2}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$3 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1} - 3}{2}, \qquad (3)$$

Solutions to Exercises on Mathematical Induction

Bagas Rahman

In Exercises 1-15 use mathematical induction to establish the formula for $n \geq 1$.

1. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof:

For $n = 1$, the statement reduces to $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$ and is obviously tru

Assuming the statement is true for $n = k$:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6},$$

we will prove that the statement must be true for $n = k + 1$:

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

The left-hand side of (2) can be written as

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2.$$

In view of (1), this simplifies to:

$$\begin{aligned}
\left(1^2 + 2^2 + 3^2 + \cdots + k^2\right) + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

Thus the left-hand side of (2) is equal to the right-hand side of (2). T proves the inductive step. Therefore, by the principle of mathemat induction, the given statement is true for every positive integer n.

2. $3 + 3^2 + 3^3 + \cdots + 3^n = \dfrac{3^{n+1} - 3}{2}$

Proof:

$3^2 - 3$

$$3+3^2+3^3+\cdots+3^k = \frac{3^{k+1}-3}{2},$$

we will prove that the statement must be true for $n = k+1$:

$$3+3^2+3^3+\cdots+3^{k+1} = \frac{3^{k+2}-3}{2}.$$

The left-hand side of (4) can be written as

$$3+3^2+3^3+\cdots+3^k+3^{k+1}.$$

In view of (3), this simplifies to:

$$\begin{aligned}(3+3^2+3^3+\cdots+3^k)+3^{k+1} &= \frac{3^{k+1}-3}{2}+3^{k+1}\\ &= \frac{3^{k+1}-3+2\cdot 3^{k+1}}{2}\\ &= \frac{3\cdot 3^{k+1}-3}{2}\\ &= \frac{3^{k+2}-3}{2}.\end{aligned}$$

Thus the left-hand side of (4) is equal to the right-hand side of (4). T proves the inductive step. Therefore, by the principle of mathemat induction, the given statement is true for every positive integer n.

3. $1^3+2^3+3^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$

Proof:

For $n=1$, the statement reduces to $1^3 = \dfrac{1^2\cdot 2^2}{4}$ and is obviously true

Assuming the statement is true for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}.$$

The left-hand side of (6) can be written as

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3.$$

In view of (5), this simplifies to:

$$\begin{aligned}\left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4}.\end{aligned}$$

Thus the left-hand side of (6) is equal to the right-hand side of (6). T proves the inductive step. Therefore, by the principle of mathemat

induction, the given statement is true for every positive integer n.

4. $1 + 3 + 6 + 10 + \cdots + \dfrac{n(n+1)}{2} = \dfrac{n(n+1)(n+2)}{6}$

Proof:
For $n = 1$, the statement reduces to $1 = \dfrac{1 \cdot 2 \cdot 3}{6}$ and is obviously tru
Assuming the statement is true for $n = k$:

$$1 + 3 + 6 + 10 + \cdots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6},$$

The left-hand side of (8) can be written as

$$1 + 3 + 6 + 10 + \cdots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}.$$

In view of (7), this simplifies to:

$$\left[1 + 3 + 6 + 10 + \cdots + \frac{k(k+1)}{2}\right] + \frac{(k+1)(k+2)}{2}$$
$$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$
$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6}$$
$$= \frac{(k+1)(k+2)(k+3)}{6}.$$

Thus the left-hand side of (8) is equal to the right-hand side of (8). T proves the inductive step. Therefore, by the principle of mathemat induction, the given statement is true for every positive integer n.

5. $1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$

Proof:

For $n = 1$, the statement reduces to $1 = \frac{1 \cdot 2}{2}$ and is obviously true. Assuming the statement is true for $n = k$:

$$1 + 4 + 7 + \cdots + (3k-2) = \frac{k(3k-1)}{2},$$

we will prove that the statement must be true for $n = k + 1$:

$$1 + 4 + 7 + \cdots + [3(k+1) - 2] = \frac{(k+1)[3(k+1) - 1]}{2}.$$